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Continuity of Effect Algebra Operations in the Interval Topology

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We study the continuity of $\oplus$ and $\ominus$ of effect algebras in the interval topology, and present several examples of effect algebras with interesting properties.

KEY WORDS: effect algebras; interval topology; continuity.

1. EFFECT ALGEBRA AND ITS ELEMENTARY PROPERTIES

In 1994, in order to model unsharp quantum logics, Foulis and Bennett introduced the following famous algebra system and called it the *effect algebras* (Foulis and Bennett, 1994):

Let L be a set with two special elements 0, 1, $\perp$ be a subset of $L \times L$. We denote $a \perp b$ if $(a, b) \in \perp$. Also, let $\oplus : \perp \to L$ be a binary operation. If the following axioms hold:

(i) (Commutative Law). If $a, b \in L$ and $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$.
(ii) (Associative Law). If $a, b, c \in L$, $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(iii) (Orthocomplementation Law). For each $a \in L$ there exists a unique $b \in L$ such that $a \perp b$ and $a \oplus b = 1$.
(iv) (Zero-Unit Law). If $a \in L$ and $1 \perp a$, then $a = 0$.

Then $(L, \perp, \oplus, 0, 1)$ is said to be an effect algebra.

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Let $(L, \perp, \oplus, 0, 1)$ be an effect algebra. If $a, b \in L$ and $a \perp b$ we say that a and b be orthogonal. If $a \oplus b = 1$ we say that b is the orthocomplement of a, and write $b = a'$. It is clear that $1' = 0$, $(a')' = a$, $a \perp 0$ and $a \oplus 0 = a$ for all $a \in L$.

We also say that $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. We may prove that $\leq$ is a partial order on L and satisfies that $0 \leq a \leq 1$, $a \leq b \Leftrightarrow b' \leq a'$ and $a \leq b' \Leftrightarrow a \perp b$ for $a, b \in L$. If $a \leq b$, the element $c \in L$ such that $c \perp a$ and $a \oplus c = b$ is unique, and satisfies the condition $c = (a \oplus b')'$. It will be denoted by $c = b \ominus a$. If $a \leq b$ but $a \neq b$, we write $a < b$.

The above showed that each effect algebra $(L, \perp, \oplus, 0, 1)$ has two binary operations $\oplus$ and $\ominus$.

If the partial order $\leq$ of an effect algebra $(L, \perp, \oplus, 0, 1)$ defined as above is a lattice, then the effect algebra $(L, \perp, \oplus, 0, 1)$ is said to be a *lattice effect algebra*; if for all $a, b \in L, a \leq b$ or $b \leq a$, then $(L, \perp, \oplus, 0, 1)$ is said to be a *totally ordered effect algebra*; if for each non-empty subset A of $(L, \perp, \oplus, 0, 1)$, the supremum $\vee\{a \in A\}$ and the infimum $\wedge\{a \in A\}$ of A exist, then $(L, \perp, \oplus, 0, 1)$ is said to be *complete*; if for all $a, b \in L$, $a < b$, there exists $c \in L$ such that $a < c < b$, then $(L, \perp, \oplus, 0, 1)$ is said to be *connected*.

Let $F = \{a_i : 1 \leq i \leq n\}$ be a finite subset of L. If $a_1 \perp a_2$, $(a_1 \oplus a_2) \perp a_3, \ldots$ and $(a_1 \oplus a_2 \ldots \oplus a_{n-1}) \perp a_n$, we say that F is *orthogonal* and we define $\oplus F = a_1 \oplus a_2 \ldots \oplus a_n = (a_1 \oplus \ldots \oplus a_{n-1}) \oplus a_n$ (by the commutative and associative laws, this sum does not depend on any permutation of elements). Now, if A is an arbitrary subset of L and $\mathcal{F}(A)$ is the family of all finite subsets of A, we say that A is *orthogonal* if F is orthogonal for each $F \in \mathcal{F}(A)$. If A is orthogonal and the supremum $\vee\{\oplus F : F \in \mathcal{F}(A)\}$ exists, then $\oplus A = \vee\{\oplus F : F \in \mathcal{F}(A)\}$ is called the $\oplus$-sum of A.

An effect algebra is said to be $\oplus$-*complete*, if for each orthogonal subsets A of L, the $\oplus$-sum $\oplus A$ exists; if for each countable orthogonal subset B of L, the $\oplus$-sum $\oplus B$ exists, then we say that the effect algebra is $\oplus$-σ-*complete*.

We may prove that each complete effect algebra must be $\oplus$-complete, but the converse is not true.

2. ORDER TOPOLOGY OF EFFECT ALGEBRAS

A partial order $(\Lambda, \preceq)$ is said to be a *directed set*, if for all $\alpha, \beta \in \Lambda$, there exists $\gamma \in \Lambda$ such that $\alpha \preceq \gamma$, $\beta \preceq \gamma$.

If $(\Lambda, \preceq)$ is a directed set and for each $\alpha \in \Lambda$, $a_\alpha \in (L, \perp, \oplus, 0, 1)$, then $\{a_\alpha\}_{\alpha \in \Lambda}$ is said to be a *net* of $(L, \perp, \oplus, 0, 1)$.

Let $\{a_\alpha\}_{\alpha \in \Lambda}$ be a net of $(L, \perp, \oplus, 0, 1)$. Then we write $a_\alpha \uparrow$, when $\alpha \preceq \beta$, $a_\alpha \leq a_\beta$. Moreover, if a is the supremum of $\{a_\alpha : \alpha \in \Lambda\}$, i.e., $a = \vee\{a_\alpha : \alpha \in \Lambda\}$, then we write $a_\alpha \uparrow a$.

Similarly, we may write $a_\alpha \downarrow$ and $a_\alpha \downarrow a$.

If $\{u_\alpha\}_{\alpha\in\Lambda}$, $\{v_\alpha\}_{\alpha\in\Lambda}$ are two nets of $(L, \perp, \oplus, 0, 1)$, for $u \uparrow u_\alpha \le v_\alpha \downarrow v$ means that $u_\alpha \le v_\alpha$ for all $\alpha \in \Lambda$ and $u_\alpha \uparrow u$ and $v_\alpha \downarrow v$. We write $b \le u_\alpha \uparrow u$ if $b \le u_\alpha$ for all $\alpha \in \Lambda$ and $u_\alpha \uparrow u$.

We say a net $\{a_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$ is *order convergent* to a point a of L if there exists two nets $\{u_\alpha\}_{\alpha\in\Lambda}$ and $\{v_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$ such that

$$a \uparrow u_\alpha \le a_\alpha \le v_\alpha \downarrow a.$$

Let $\mathcal{F} = \{F : F = \emptyset$ or $F \subseteq L$ and satisfies that for each net $\{a_\alpha\}_{\alpha\in\Lambda}$ of F if $\{a_\alpha\}_{\alpha\in\Lambda}$ is order convergent to a, then $a \in F\}$.

It is easy to prove that $\emptyset$, $L \in \mathcal{F}$ and if $F_1, F_2, \ldots, F_n \in \mathcal{F}$, then $\cup_{i=1}^n F_i \in \mathcal{F}$, if $\{F_\mu\}_{\mu\in\Omega} \subseteq \mathcal{F}$, then $\cap_{\mu\in\Omega} F_\mu \in \mathcal{F}$. Thus, the family $\mathcal{F}$ of subsets of L define a topology τ_0^L on $(L, \perp, \oplus, 0, 1)$ such that $\mathcal{F}$ consists of all closed sets of this topology. The topology τ_0^L is called the *order topology* of $(L, \perp, \oplus, 0, 1)$ (Birkhoff, 1948).

We can prove that the order topology τ_0^L of $(L, \perp, \oplus, 0, 1)$ is the finest (strongest) topology on L such that for each net $\{a_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$, if $\{a_\alpha\}_{\alpha\in\Lambda}$ is order convergent to a, then $\{a_\alpha\}_{\alpha\in\Lambda}$ must be topology τ_0^L convergent to a. But the converse is not true.

For the order convergent properties of nets in effect algebras, Riecanova proved the following conclusions (Riecanova, 2000):

Lemma 2.1. *Let $(L, \perp, \oplus, 0, 1)$ be a lattice effect algebra. For elements of L we have:*

(1) *$b' \ge a_\alpha \downarrow a$ implies that $a_\alpha \oplus b \downarrow a \oplus b$.*
(2) *$b \le a_\alpha \uparrow a$ implies that $a_\alpha \ominus b \uparrow a \ominus b$.*
(3) *$b' \ge a_\alpha$ order convergent to a implies that $a_\alpha \oplus b$ order convergent to $a \oplus b$.*
(4) *$b \le a_\alpha$ order convergent to a implies that $a_\alpha \ominus b$ order convergent to $a \ominus b$.*
(5) *$b' \ge a_\alpha$ order convergent to a iff $a_\alpha \oplus b$ order convergent to $a \oplus b$.*
(6) *$b \le a_\alpha$ order convergent to a iff $a_\alpha \ominus b$ order convergent to $a \oplus b$.*
(7) *$b \ge a_\alpha$ order convergent to a iff $b \ominus a_\alpha$ order convergent to $b \ominus a$.*

Furthermore, after proving two Lemmas, Riecanova proved the continuity of $\oplus$ and $\ominus$ with respect to the order topology, that is:

Theorem 2.1. *If $(L, \perp, \oplus, 0, 1)$ is a lattice effect algebra, then a net $\{a_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$ has:*

(1) *If $b' \ge a_\alpha$ for all $\alpha \in \Lambda$, and $\{a_\alpha\}_{\alpha\in\Lambda}$ convergent to a with respect to the order topology τ_0^L, then $\{a_\alpha \oplus b\}$ convergent to $a \oplus b$ with respect to the order topology τ_0^L.*

(2) *If* $b \leq a_\alpha$ *for all* $\alpha \in \Lambda$, *and* $\{a_\alpha\}$ *convergent to* a *with respect to the order topology* τ_0^L, *then* $\{a_\alpha \ominus b\}$ *convergent to* $a \ominus b$ *with respect to the order topology* τ_0^L.

(3) *If* $b \geq a_\alpha$ *for all* $\alpha \in \Lambda$, *and* $\{a_\alpha\}$ *convergent to* a *with respect to the order topology* τ_0^L, *then* $\{b \ominus a_\alpha\}$ *convergent to* $b \ominus a$ *with respect to the order topology* τ_0^L.

3. INTERVAL TOPOLOGY OF EFFECT ALGEBRAS

A family of closed sets $\mathcal{F}$ in a topological space (X, T) with the property that each closed subset of (X, T) is an intersection of members of $\mathcal{F}$, is called a *basis* of closed sets.

A *sub-basis* of closed sets is a family $\mathcal{S}$ of closed sets, such that each set of some basis is the union of finite sets of $\mathcal{S}$, hence each closed set is an intersection of finite unions of sets of $\mathcal{S}$.

Definition 3.1. By the interval topology of an effect algebra $(L, \perp, \oplus, 0, 1)$, we mean that defined by taking the all closed intervals $[a, b]$ as a sub-basis of closed sets of $(L, \perp, \oplus, 0, 1)$.

It is easy to prove that each closed interval $[a, b]$ of effect algebra $(L, \perp, \oplus, 0, 1)$ is a closed set with respect to the order topology of effect algebra, so the interval topology is weaker than the order topology. But, if $(L, \perp, \oplus, 0, 1)$ is a totally order effect algebra, then its order topology and interval topology are same.

Now, we present an example of effect algebra to show that its interval topology may be really weaker than its order topology.

Example 3.1. Let $L = \{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$. For each $\frac{1}{n}$, let $0 \oplus \frac{1}{n} = \frac{1}{n}$, $\frac{1}{n} \oplus \frac{1}{n} = 1$, $0 \oplus 1 = 1$. If $m \neq n$, $\frac{1}{m} \oplus \frac{1}{n}$ can not be defined. Then $(L, 0, 1, \oplus)$ is a complete effect algebra, and the interval topology is really weaker than the order topology.

Now, we need the following lemma which is a famous fact in the topology theory:

Lemma 3.1. *If* (X, T_1) *and* (Y, T_2) *are two topological spaces and* $f : (X, T_1) \to (Y, T_2)$. *Then* f *is a continuous map iff for each closed subset* A *of* (Y, T_2), *the inverse image* $f^{-1}(A)$ *of* A *is a closed subset of* (X, T_1).

Now, we study the continuity of $\oplus$ and $\ominus$ of effect algebras with respect to the interval topology, our main result is:

Theorem 3.1. *If* $(L, \perp, \oplus, 0, 1)$ *is a lattice effect algebra, then a net* $\{a_\alpha\}_{\alpha\in\Lambda}$ *of* $(L, \perp, \oplus, 0, 1)$ *has:*

(1) *If* $b' \geq a_\alpha$ *for all* $\alpha \in \Lambda$ *and* $\{a_\alpha\}_{\alpha\in\Lambda}$ *convergent to a with respect to the interval topology, then* $\{a_\alpha \oplus b\}$ *convergent to* $a \oplus b$ *with respect to the interval topology.*

(2) *If* $b \leq a_\alpha$ *for all* $\alpha \in \Lambda$ *and* $\{a_\alpha\}$ *convergent to a with respect to the interval topology, then* $\{a_\alpha \ominus b\}$ *convergent to* $a \ominus b$ *with respect to the interval topology.*

(3) *If* $b \geq a_\alpha$ *for all* $\alpha \in \Lambda$ *and* $\{a_\alpha\}$ *convergent to a with respect to the interval topology, then* $\{b \ominus a_\alpha\}$ *convergent to* $b \ominus a$ *with respect to the interval topology.*

Proof: We only prove conclusion (1), the conclusions (2) and (3) can be proved by the similar methods.

It follows from the definition of interval topology and Lemma 3.1 that we only need to prove for each closed interval $[c, d]$ of $(L, \perp, \oplus, 0, 1)$, the inverse image $I = \{x : x \leq b' andx \oplus b \in [c, d]$ is a closed set of $(L, \perp, \oplus, 0, 1)$ with respect to the interval topology.

(i) If $[c, d]$ only has an element or $[c, d]$ is a empty set, then it is clear that I only has an element or I is an empty set.

(ii) If $b \leq c \leq d$, then $I = [c \ominus b, d \ominus b] \cap [0, b']$.

(iii) If $c < d$ and $d \leq b$, then I is an empty set.

(iv) If $c \leq b \leq d$, then $I = [0, d \ominus b] \cap [0, b']$.

(v) If b cannot be compared with c, b cannot also be compared with d, then I is an empty set.

(vi) If b cannot be compared with c, but $b \leq d$. Let $e = c \vee b$. Then $I = [e \ominus b, d \ominus b] \cap [0, b']$.

(vii) If b cannot be compared with d, but $c \leq b$. Let $f = b \wedge d$. Then $I = [0, f \ominus b] \cap [0, b']$.

It follows from (i)–(vii) that conclusion (1) is true and the theorem is proved. □

It is well known that if the effect algebra is complete, then its interval topology is a compact topological space (Birkhoff, 1948). So the Example 1 is a compact topological space if its topology is the interval topology.

Example 3.2. Let $E = [0, 1]$, $x, y \in E$ and $x \oplus y$ be defined iff $x + y \leq 1$. Then $(E, 0, 1, \oplus)$ is a compact totally ordered connected effect algebra.

In 1995, Mesiar showed that an effect algebra is a totally ordered compact and connected iff it is isomorphic with the effect algebra $(E, 0, 1, \oplus)$ of Example 3.2 (Mesiar, 1995).

Now, we present a connected effect algebra, but it is not complete and totally ordered.

Example 3.3. Let $L = [0, 1] \times [0, 1]$, $(x_1, x_2), (y_1, y_2) \in L$ and $(x_1, x_2) \oplus (y_1, y_2)$ be defined iff $x_1 + y_1 \leq 1, x_2 + y_2 \leq 1$. Then $(L, 0, 1, \oplus)$ is connected, but it is not complete and it is not also totally ordered. In addition, the open interval ([0, 0], [1, 0]) is not an open set with respect to its interval topology.

Furthermore, we present an example which is a totally ordered connected effect algebra, but it is not $\oplus$-σ-complete and it is not also compact with respect to the interval topology.

Example 3.4. Let L be the all rational numbers of [0, 1], $x, y \in L$, and $x \oplus y$ be defined iff $x + y \leq 1$. Then $(L, 0, 1, \oplus)$ is a totally ordered connected effect algebra, but it is not $\oplus$-σ-complete and it is not also compact effect algebra with respect to the interval topology.

Note that $\{\frac{1}{n}\}$ in Example 1 is convergent to 1 with respect to the interval topology, but $\frac{1}{n} \oplus \frac{1}{n} = 1$ is not convergent to $1 \oplus 1$, so the $\oplus$ operation is not continuous for two variables.

Finally, we prove the following interesting conclusion:

Theorem 3.2. *Let $(L, \perp, \oplus, 0, 1)$ be a lattice effect algebra. If $A = \{a_k\}_{k \in \mathbf{N}}$ is orthogonal $\oplus$-summable, then $\{a_n\}_{n \in \mathbf{N}}$ is order convergent to 0, and so $\{a_n\}_{n \in \mathbf{N}}$ is also order topology and interval topology convergent to 0.*

Note that if $\{a_n\}_{n \in \mathbf{N}}$ is order convergent to 0, then $\{a_n\}_{n \in \mathbf{N}}$ must be order topology convergent to 0, and interval topology is weaker than the order topology, so we only need to prove $\{a_n\}_{n \in \mathbf{N}}$ is order convergent to 0.

In fact, let $a = \oplus A = \vee\{\oplus_{k=1}^{n} a_k : n \in \mathbf{N}\}$ and $s_n = \vee\{\oplus_{k=1}^{n} a_k\}$. Then it follows from Lemma 2.1 that $0 \leq a_n \leq a \ominus s_{k+1}$ and $a \ominus s_{k+1} \downarrow 0$. So $\{a_k\}$ is order convergent to 0.

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